Topics in Database Theory – Homework 1

1 The Satisfiability Problem

1. (0 points)

   (a) Recall that, by definition, every structure has a non-empty domain. Prove that the following sentence is valid:

   $$\varphi = \exists x (U(x) \Rightarrow \forall z U(z))$$

   (b) For each of the sentences below indicate whether it is satisfiable or not. If it is satisfiable, given an example of a structure that satisfies that sentence.

   $$\varphi_1 = \exists x \forall y \, (U(y) \Rightarrow x = y)$$

   $$\varphi_2 = \exists x_1 \exists x_2 \, ((x_1 \neq x_2) \wedge V(x_1) \wedge V(x_2))$$

   $$\varphi_3 = \varphi_1 \wedge \varphi_2 \wedge \forall z \, (U(z) \Rightarrow V(z))$$

   $$\varphi_4 = \varphi_1 \wedge \varphi_2 \wedge \forall z \, (V(z) \Rightarrow U(z))$$

   $$\begin{aligned} \varphi_5 = &\forall x \, (U(x) \rightarrow (V(x) \vee W(x))) \\ &\wedge \forall y \, (V(y) \rightarrow (U(y) \vee W(y))) \\ &\wedge \forall z \, (W(z) \rightarrow (U(z) \vee V(z))) \\ &\wedge \neg \exists x (U(x) \wedge V(x) \wedge W(x)) \end{aligned}$$

(c) Prove that, if the vocabulary is empty, then satisfiability is decidable, and that finite satisfiability coincides with general satisfiability. Notice that, when the vocabulary is empty, then we can only use the symbols $=, \exists, \forall, \vee, \wedge, \neg$. For example, the following is a sentence in this language, saying that the domain has at least 3 elements:

$$\varphi = \forall x \forall y \exists z (z \neq x \wedge z \neq y)$$

Hint: what can such a sentence say about the domain D? The sentence above says that $|D| \geq 3$. What else can you say in this language? Once you figure that out and prove it, observe that φ is satisfiable iff it is finitely satisfiable, then use this to prove decidability.

(d) (Challenging) Prove that if the vocabulary is unary, i.e. $\sigma = \{U_1, \ldots, U_m\}$ where each U_i is a unary predicate, then satisfiability is decidable and, moreover, general satisfiability coincides with finite satisfiability. Notice that this does not contradict Trakhtenbrot's theorem (why?). An example of a sentence in this language is the following:

$$\varphi = \forall x (U_1(x) \Rightarrow \exists y (U_1(y) \wedge x \neq y \wedge \neg U_2(y)))$$

(What does it say?).

Hint: you can extend your proof from the previous question. This is instructive, because you understand how to analyze FO sentences. As before, a much shorter proof uses Ehrenfeucht–Fraïssé games [1]. Using EF-games, you can prove easily that, if φ has any model $\boldsymbol{A}$, then it also has a model whose domain has size $\leq k2^m$, where k is the number of variables in φ.

2 Domain Independence

2. (0 points)

(a) Indicate for each query below whether it is domain independent and, if it is, then provide an equivalent Relational Algebra expression, i.e. draw a query plan.

$$\begin{aligned}
Q_1(X) &= \forall Y (S(Y) \wedge R(X, Y)) \\
Q_2(X) &= T(X) \wedge \forall Y (S(Y) \Rightarrow R(X, Y)) \\
Q_3(X) &= T(X) \wedge \forall Y (R(X, Y) \Rightarrow S(Y)) \\
Q_4(X) &= T(X) \wedge \forall Y (R(X, Y))
\end{aligned}$$

References

[1] L. Libkin. *Elements of Finite Model Theory.* Texts in Theoretical Computer Science. An EATCS Series. Springer, 2004.